IITians GATE CLASSES
Exclusive GATE Coaching by IIT/IISc Graduates
A division of PhIE Learning Center

IITians GATE CLASSES
BANGALORE
Visit us: www.iitiansgateclasses.com
Mail us: info@iitiansgateclasses.com

IITians GATE CLASSES
Exclusive GATE Coaching by IIT/IISc Graduates
A division of PhIE Learning Center

Mechanical Vibration
(GATE Aerospace and GATE Mechanical) by
Mr Dinesh Kumar (IIT Madras Fellow)

Damped free vibration of single degree of freedom

When damped free vibration take place, the amplitude of vibration gradually becomes small and finally is completely lost.

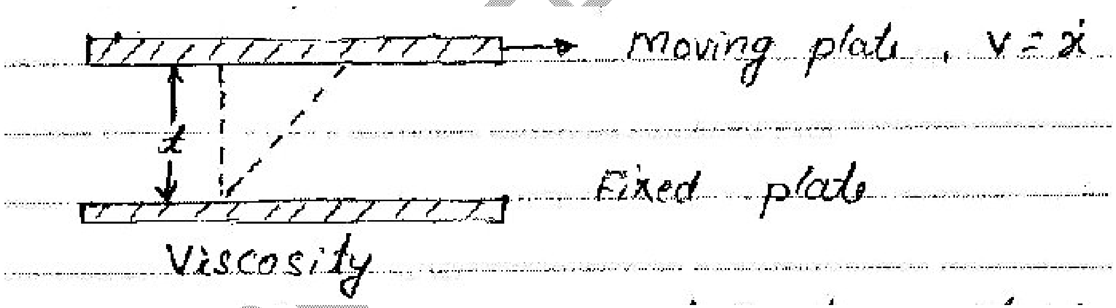
The rate, at which the amplitude decays, depends upon the type and amount of damping in the system.

- **Different type of damping**

- **Viscous damping**

This is the most important type of damping and occurs formally with velocities in lubricated sliding surface, dashpots with small clearance etc.

Eddy current damping is also type of viscous nature. The amount of resistance will depend upon the relative velocity and upon the parameter of the damping system.



Consider two plates are separated by fluid film of thickness t. The upper plate is allowed to move parallel to the fixed plate with a velocity $\dot{x}$. The net force F required for maintaining the velocity $\dot{x}$ of the plate is expressed as;

$$F = \frac{\mu A}{t}\,\dot{x}$$

Where, A = Area of the plate

t = Thickness

μ = Coefficient of absolute viscosity of the film

$$F = c\,\dot{x}$$

Where,

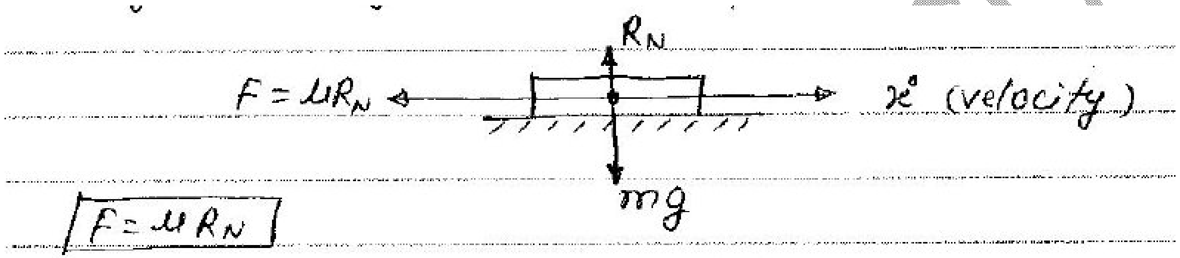
$$c = \frac{\mu A}{t}$$

Where, c is viscous damping coefficient.

- **Dry friction or coulomb damping**

This type of damping occurs when two machine parts rub against each other, dry or un- lubricated. The damping resistance in the case is practically constant and is independent of the rubbing velocity.

General expression for coulomb damping is:



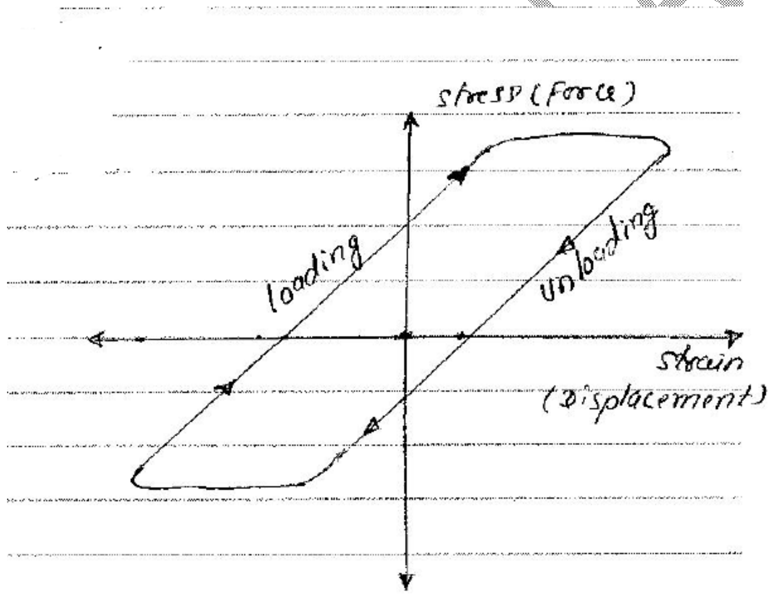
Where,

μ = coefficient of friction

R_N = Normal reaction

- **Structural damping**

This type of damping is due to the internal friction of the molecules. The stress – strain diagram for vibrating body is not a straight line but forms a hysteresis loop. The area of which represents the energy dissipated due to molecular friction per cycle per unit volume.



The energy loss per cycle,

E = $\pi k \lambda A^2$

Where,

A = amplitude of vibration

λ = dimensionless damping factor

E = βA^2

Where,

$\beta = \pi k \lambda$

- **Energy dissipation in viscous damping**

For a viscously damped system the force,

F = c$\dot{x}$ = c$\frac{dx}{dt}$

Work done, dW = F.dx = (c$\frac{dx}{dt}$)dx

The rate of change of work per cycle i.e. energy dissipated

$\Delta E = \int_0^{2\pi/\omega} (F.\frac{dx}{dt})$ dt = $\int_0^{2\pi/\omega} ((c\frac{dx}{dt}).\frac{dx}{dt})$ dt

$\Delta E = \int_0^{2\pi/\omega} c.(\frac{dx}{dt})^2$ dt

If response is simple harmonic,

x = Asin ωt

$(\frac{dx}{dt})^2 = \omega^2 A^2 (\cos \omega t)^2$

$\Delta E = \int_0^{2\pi/\omega} c.\omega^2 A^2 (\frac{1+\cos 2\omega t}{2})$ dt

$\Delta E = \pi c \omega A^2$

The max kinetic energy of the system,

E = $(K.E.)_{max} = \frac{1}{2} m \dot{x}_{max}^2 = \frac{1}{2} m \omega^2 A^2$

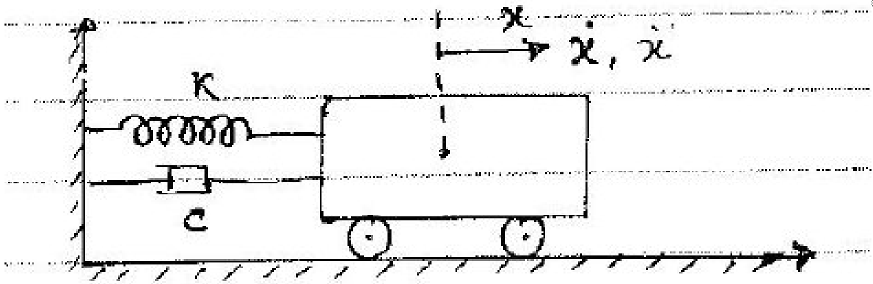
The ratio of $\Delta E / E$,

$\beta = \frac{\Delta E}{E} = \frac{\pi c \omega A^2}{\frac{1}{2} m \omega^2 A^2}$

$\beta = \frac{2c\pi}{m\omega}$

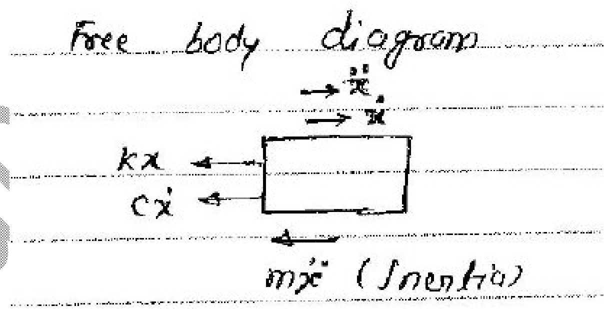
Where β is constant and known as specific damping capacity of the system.

- **Differential equation of damped free vibration**



The equation of motion,

$m\ddot{x} + c\dot{x} + kx = 0$



Above equation is a linear differential equation of 2nd order and solution can be written as,

x = e^{st}

Then

$ms^2e^{st} + cse^{st} + ke^{st} = 0$

$(ms^2 + cs + k)\, e^{st} = 0$

$ms^2 + cs + k = 0$

The above equation is called characteristic equation of the systems,

$$\mathbf{S_{1,2}} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

General solution for free damped vibration of single degree,

x = $c_1\, e^{S_1 t} + c_2\, e^{S_2 t}$

Where c_1 and c_2 are constants to be determined from initial condition.

Critical damped coefficient

If roots of characteristic equation are real and equal then the value of damping coefficient is known as critical damping coefficient, i.e.

$$\left(\frac{C_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\left(\frac{C_c}{2m}\right)^2 = \frac{k}{m}$$

$$\frac{C_c}{2m} = \sqrt{\frac{k}{m}}$$

$$\frac{C_c}{2m} = \omega_n$$

$$C_c = 2m\omega_n = 2\sqrt{km}$$

Damping factor or damping ratio

The ratio of damping coefficient to critical damping coefficient is known as damping factor or damping ratio.

$\zeta = \frac{C}{C_c}$

Now, $\frac{C}{2m} = \frac{C}{C_c} \cdot \frac{C_c}{2m} = \zeta\omega_n$

So,

$$\mathbf{S_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\,\omega_n}$$

If C > C_c or ζ > 1, system is over damped

C = C_c or ζ = 1, system is critically- damped

C < C_c or ζ < 1, system is under-damped

Over damped system ($\zeta > 1$)

General solution,

$$x = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

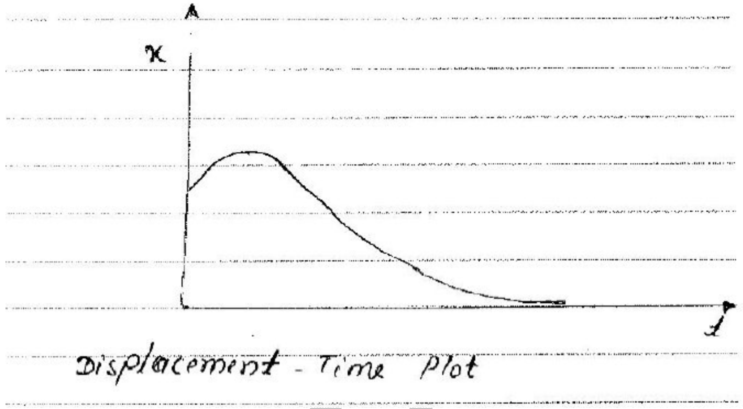
For critical condition, x = X_0

And $\dot{x}$ = 0

At t = 0

Solving for the condition,

$$x = \frac{X_0}{2\sqrt{\zeta^2-1}}\left[\left(-\zeta + \sqrt{\zeta^2 - 1}\right) e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + \left(-\zeta - \sqrt{\zeta^2 - 1}\right) e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}\right]$$



Critical damped system (ζ = 1)

$S_1 = S_2 = -\omega_n$

Solution,

$x = (C_1 + C_2 t)e^{-\omega_n t}$

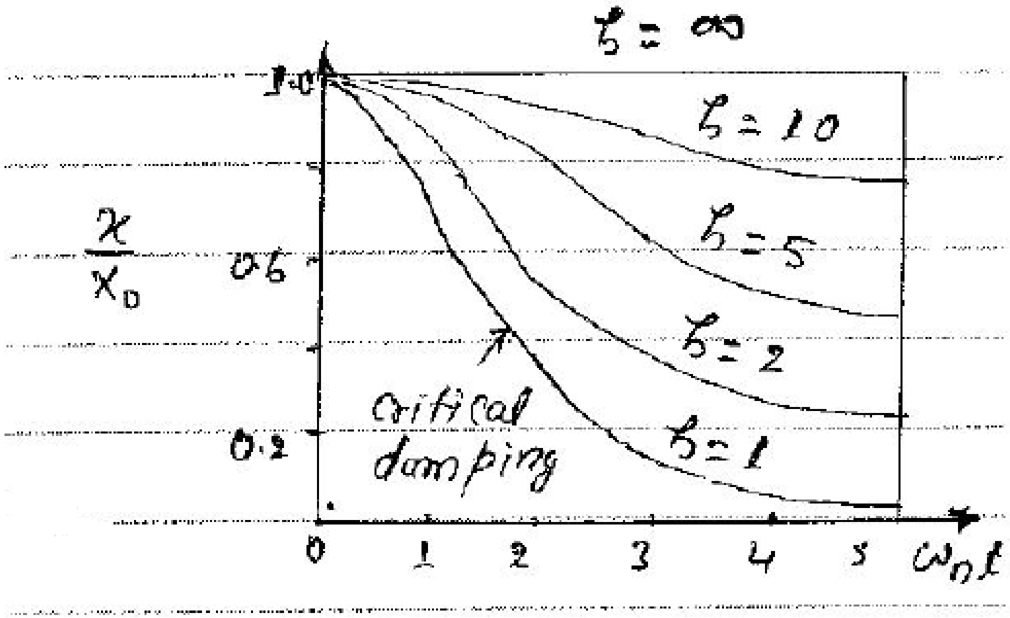
For initial condition,

$x = X_0$

And $\dot{x} = 0$

At t = 0

$\rightarrow x = X_0(1 + \omega_n t)\, e^{-\omega_n t}$



Displacement - Time plot

these curves the system comes to the equilibrium position in nearly an exponential manner. Higher the damping, more sluggish is the response of the system. This may even be seen physically that higher the damping, more resistance to motion is there; and therefore slower is the movement. Theoretically, however, the system will take infinite time to come back to the equilibrium position once it is disturbed from it. This type of motion is called *aperiodic motion*.

- Critical damping is smallest damping for which response is non-oscillatory
- With Critical damping system will have steep fall in displacement

Under – damped system (ζ < 1)

For under – damped system, roots of characteristic equation,

$S_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\,\omega_n$

$S_{1,2} = (-\zeta \pm i\sqrt{1 - \zeta^2})\,\omega_n$

General solution,

$x = e^{-\zeta\omega_n t}\,[C_1 e^{i(\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{-i(\sqrt{1-\zeta^2})\omega_n t}]$

$(\sqrt{1 - \zeta^2})\omega_n = \omega_d \quad (\omega_d = damped\ natural\ frequency)$

For initial condition,

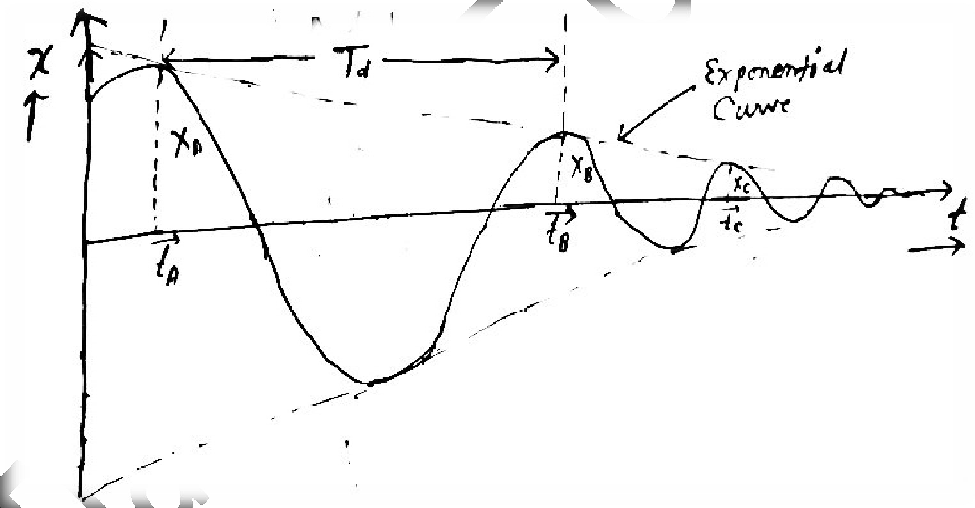
$\left.\begin{array}{l} x = X_0 \\ \text{And} \quad \dot{x} = 0 \end{array}\right\} \rightarrow \boxed{\text{At} \quad t = 0}$

$x = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \emptyset)$

Where, $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Amplitude = $\frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$

Which is seems to decay exponentially with time. Theoretically, the system will never come to rest, although the amplitude of vibration may become infinitely small.



Period, $T_d = \frac{2\pi}{\omega_d} \quad = \frac{2\pi}{\omega_n(\sqrt{1-\zeta^2})}$

- **Logarithmic decay**

It is the ratio of any two successive amplitudes for an under-damped system vibrating freely is constant and it is a function of damping only.

Consider two points A and B, corresponding to the times t_A and t_B.

$$t_B - t_A = \frac{2\pi}{\omega_d} \quad = \frac{2\pi}{\omega_n(\sqrt{1-\zeta^2})} = T_d$$

Amplitude of oscillation for under-damped system is

$$\text{Amplitude} = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$$

Therefore,

$$X_A = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_A}$$

$$X_B = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_B}$$

$$\frac{X_A}{X_B} = e^{\zeta\omega_n(t_B - t_A)} = e^{\zeta\omega_n t_d}$$

$$\frac{X_A}{X_B} = e^{\zeta\omega_n \frac{2\pi}{\omega_n(\sqrt{1-\zeta^2})}} = e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$\log\frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

This is called logarithmic decay, is denoted by δ

$$\delta = \ln\frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\delta = \ln\frac{X_0}{X_1} = \ln\frac{X_1}{X_2} = \ln\frac{X_2}{X_3} = \text{..........} \ln\frac{X_{n-1}}{X_n}$$

$$n\delta = \ln\frac{X_0}{X_1} + \ln\frac{X_1}{X_2} + \ln\frac{X_2}{X_3} + \text{..........} \ln\frac{X_{n-1}}{X_n}$$

$$n\delta = \ln\frac{X_0}{X_1}.\frac{X_1}{X_2}.\frac{X_2}{X_3}.....\frac{X_{n-1}}{X_n}$$

$$n\delta = \ln\frac{X_0}{X_n}$$

$\rightarrow \delta = \frac{1}{n} \ln \frac{X_0}{X_n}$

Here, amplitude = n+1

Cycles = n - 0 = n

- **Amplitude decay in coulomb's damping**

Case 1. When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton's second law (see Fig. 2.42(b)):

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N \tag{2.126}$$

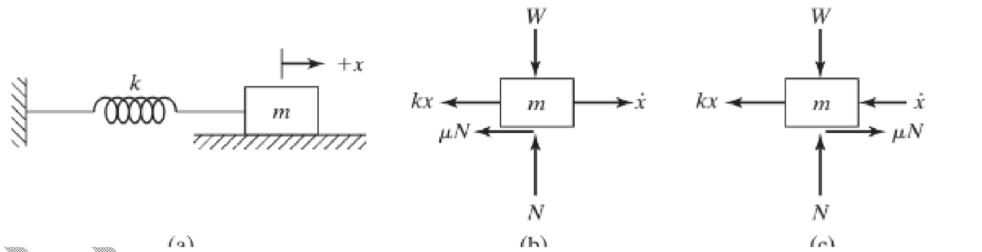
This is a second-order nonhomogeneous differential equation. The solution can be verified by substituting Eq. (2.127) into Eq. (2.126):

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k} \tag{2.127}$$

where $\omega_n = \sqrt{k/m}$ is the frequency of vibration and A_1 and A_2 are constants whose values depend on the initial conditions of this half cycle.

Case 2. When x is positive and dx/dt is negative or when x is negative and dx/dt is negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion can be derived from Fig. 2.42(c) as

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N \tag{2.128}$$



Note the following characteristics of a system with Coulomb damping:

1. The equation of motion is nonlinear with Coulomb damping, whereas it is linear with viscous damping.
2. The natural frequency of the system is unaltered with the addition of Coulomb damping, whereas it is reduced with the addition of viscous damping.
3. The motion is periodic with Coulomb damping, whereas it can be nonperiodic in a viscously damped (overdamped) system.
4. The system comes to rest after some time with Coulomb damping, whereas the motion theoretically continues forever (perhaps with an infinitesimally small amplitude) with viscous and hysteresis damping.
5. The amplitude reduces linearly with Coulomb damping, whereas it reduces exponentially with viscous damping.
6. In each successive cycle, the amplitude of motion is reduced by the amount $4\mu N/k$, so the amplitudes at the end of any two consecutive cycles are related:

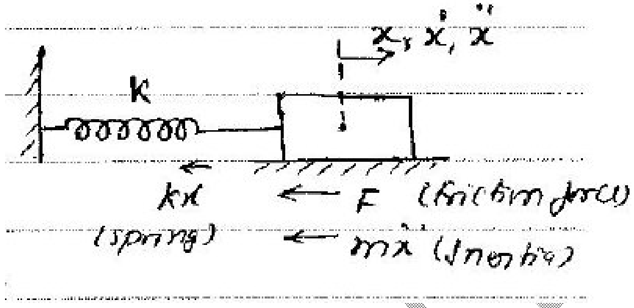
$$X_m = X_{m-1} - \frac{4\mu N}{k} \tag{2.135}$$

These become the initial conditions for the third half cycle, and the procedure can be continued until the motion stops. The motion stops when $x_n \leq \mu N/k$, since the restoring force exerted by the spring (kx) will then be less than the friction force μN. Thus the number of half cycles (r) that elapse before the motion ceases is given by

$$x_0 - r\frac{2\mu N}{k} \leq \frac{\mu N}{k}$$

that is,

$$r \geq \left\{ \frac{x_0 - \dfrac{\mu N}{k}}{\dfrac{2\mu N}{k}} \right\} \tag{2.134}$$



Here Mass is moving from LHS to RHS and initial displacement is -X_0

Equation for motion

m$\ddot{x}$ + kx + F = 0

Solution,

x = Acos($\sqrt{k/m})t$ + Bsin($\sqrt{k/m})t$ - F/k

Initial condition,

$$\left.\begin{array}{l} x = -X_0 \\ \dot{x} = 0 \end{array}\right\} \rightarrow \boxed{\text{At } t = 0}$$

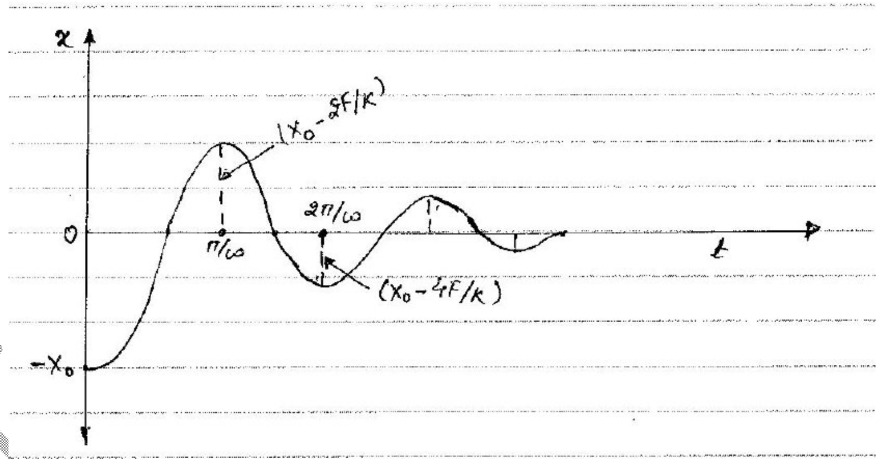
This gives A = ($-X_0 + F/k$), B= 0

x = $(-X_0 + F/k)\cos(\sqrt{k/m})t$ - F/k

This solution holds good for half cycle, when t = $\frac{\pi}{\omega_n}$ half cycle is complete. Displacement for half cycle

x = $(-X_0 + F/k)\cos\pi$ + F/k = $X_0 - 2F/k$

The critical displacement X_0 is redundant by 2F/k in half cycle and in full cycle, it is redundant by 4F/k.



1.

A spring-mass system is viscously damped with a viscous damping constant c. The energy dissipated per cycle when the system is undergoing a harmonic vibration $XCos\omega_d t$ is given by

(A) $\pi c\omega_d X^2$ (B) $\pi\omega_d X^2$ (C) $\pi c\omega_d X$ (D) $\pi c\omega_d^2 X$

[AE GATE 2012]

2.

Critical damping is the

(A) largest amount of damping for which no oscillation occurs in free vibration
(B) smallest amount of damping for which no oscillation occurs in free vibration
(C) largest amount of damping for which the motion is simple harmonic in free vibration
(D) smallest amount of damping for which the motion is simple harmonic in free vibration

[ME GATE 2014]

3.

A single degree of freedom mass-spring-viscous damper system with mass m, spring constant k and viscous damping coefficient q is critically damped. The correct relation among m, k, and q is

(A) $q = \sqrt{2km}$ (B) $q = 2\sqrt{km}$ (C) $q = \sqrt{\frac{2k}{m}}$ (D) $q = 2\sqrt{\frac{k}{m}}$

[ME GATE 2016]

4.

The equation of motion of a harmonic oscillator is given by

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0,$$

and the initial conditions at $t = 0$ are $x(0) = X, \frac{dx}{dt}(0) = 0$. The amplitude of $x(t)$ after n complete cycles is

(A) $Xe^{-2n\pi\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}$

(B) $Xe^{2n\pi\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}$

(C) $Xe^{-2n\pi\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$

(D) X

[ME GATE 2007]

5.

The damping ratio of a single degree of freedom spring-mass-damper system with mass of 1 kg, stiffness 100 N/m and viscous damping coefficient of 25 N.s/m is ________

[ME GATE 2014]

6.

A spring-mass-damper system with a mass of 1 kg is found to have a damping ratio of 0.2 and a natural frequency of 5 rad/s. The damping of the system is given by

(A) 2 Ns/m (B) 2 N/s (C) 0.2 kg/s (D) 0.2 N/s

[AE GATE 2007]

7.

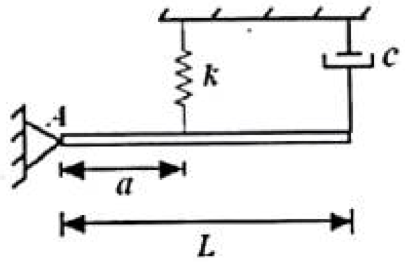
In a spring-mass-damper single degree of freedom system, the mass is 2 kg and the undamped natural frequency is 20 Hz. The critical damping constant of the system is

(A) 160π N.s/m (B) 80π N.s/m (C) 1 N.s/m (D) 0 N.s/m

[AE GATE 2009]

8.

A uniform rigid bar of mass $m = 1$ kg and length $L = 1$ m is pivoted at A. It is supported by a spring of stiffness $k = 1$ N/m and a viscous damper of damping constant $C = 1$ N-s/m, with $a = 1/\sqrt{3}$ m as shown below. The moment of inertia of the rigid bar is $I_A = \dfrac{mL^2}{3}$.



The system is

(A) overdamped
(B) underdamped with natural frequency $\omega_n = 1$ rad/s
(C) critically damped
(D) underdamped with natural frequency $\omega_n = 2$ rad/s

[AE GATE 2009]

9.

During an under-damped oscillation of a single degree of freedom system, in the time-displacement plot the third peak is of magnitude 100 and the tenth peak is of magnitude 10. The damping ratio ζ is approximately:

(A) 0.052 (B) 0.023 (C) 0.366 (D) 0.159

[AE GATE 2010]

10.

Consider a single degree of freedom spring-mass-damper system with mass, damping and stiffness of m, c and k, respectively. The logarithmic decrement of this system can be calculated using

(A) $\frac{2\pi c}{\sqrt{4mk-c^2}}$ (B) $\frac{\pi c}{\sqrt{4mk-c^2}}$ (C) $\frac{2\pi c}{\sqrt{mk-c^2}}$ (D) $\frac{2\pi c}{\sqrt{mk-4c^2}}$

[AE GATE 2011]

11.

The logarithmic decrement measured for a viscously damped single degree of freedom system is 0.125. The value of the damping factor in % is closest to

(A) 0.5 (B) 1.0 (C) 1.5 (D) 2.0

[AE GATE 2012]

12.

A linear mass-spring-dashpot system is over-damped. In free vibration, this system undergoes

(A) non-oscillatory motion
(B) random motion
(C) oscillatory and periodic motion
(D) oscillatory and non-periodic motion

[AE GATE 2016]

13.

A single degree of freedom spring-mass system of natural frequency 5 Hz is modified in the following manners:
Case 1: Viscous damping with damping ratio $\varsigma = 0.2$ is introduced in parallel to the spring.
Case 2: The original undamped spring-mass system is moved to a surface with coefficient of friction, $\mu = 0.01$.
The ratio of the damped natural frequency for the cases 1 and 2 is given by _________(in three decimal places).

[AE GATE 2017]

14.

An aircraft landing gear can be idealized as a single degree of freedom spring-mass-damper system. The desirable damping characteristics of such a system is:

(A) Under damped
(B) Over damped
(C) Critically damped
(D) Undamped

[AE GATE 2017]

15.

A single degree of freedom vibrating system has mass of 5 kg, stiffness of 500 N/m and damping coefficient of 100 N-s/m. To make the system critically damped

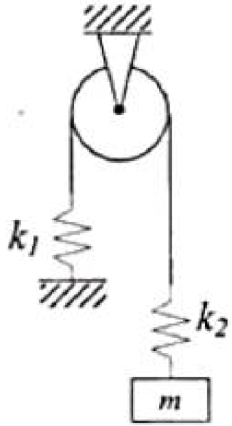
(A) only the mass is to be increased by 1.2 times.
(B) only the stiffness is to be reduced to half.
(C) only the damping coefficient is to be doubled.
(D) no change in any of the system parameters is required.

[XE GATE 2016]

16.

(32)

A single degree freedom system consisting of 2 springs and a mass is shown in the figure.



The natural frequency of the system in radians/sec is given by

(A) $\sqrt{\frac{k_1}{m}}$ (B) $\sqrt{\frac{k_2}{m}}$ (C) $\sqrt{\frac{k_1 k_2}{(k_1+k_2)m}}$ (D) $\sqrt{\frac{k_1+k_2}{m}}$

[XE GATE 2007]

17.

For a damped spring-mass system, mass $m = 10$ kg, stiffness $k = 10^3$ N/m, and damping coefficient $c = 20$ kg/s. The ratio of the amplitude of oscillation of the first cycle to that of the fifth cycle is ______ *(round off to 1 decimal place)*.

18.

A 1 m long massless cantilever beam oscillates at 2Hz, while a 60 kg mass is attached at the tip of it. The flexural rigidity of the beam (in kN-m^2) is ________________ (accurate to two decimal places).

(13)

Logarithmic decrement of a damped single degree of freedom system is δ. If stiffness of the spring is doubled and mass is made half, then logarithmic decrement of the new system will be equal to

(a) $1/2\delta$	(b) δ
(c) 2δ	(d) ¼ δ

[ME ISRO 2013]

(14)

The equation of motion for a damped vibration is given by $6\ddot{x} + 9\dot{x} + 27x = 0$. The damping factor will be

(a) 0.25	(b) 0.5	(c) 0.35	(d) 0.75

[ME ISRO 2013]

(15)

In the viscous damped vibration, the logarithmic decrement value over five cycles is found to be 8.11. What is viscous damping factor of vibratory system?

(a) 20% (b) 25% (c) 30% (d) 15%

[ME ISRO 2013]